To prove that Gaussian elimination can be performed without row interchanges, we show that each of the matrices $A^{(2)}, A^{(3)}, \ldots, A^{(n)}$ is strictly diagonally dominant. (at each step pivot elements are nonzero)

Since A is strictly diagonally dominant, $a_{11} \neq 0$ and $A^{(2)}$ can be formed

$$a_{ij}^{(2)} = a_{ij}^{(1)} - \frac{a_{1j}^{(1)} a_{i1}^{(1)}}{a_{11}^{(1)}}, \quad \text{for} \quad 2 \leq j \leq n$$

$$i = 2, 3, \ldots, n,$$

First, $a_{i1}^{(2)} = 0$. The triangle inequality implies that

$$\sum_{\substack{j=2 \\ j \neq i}}^{n} |a_{ij}^{(2)}| = \sum_{\substack{j=2 \\ j \neq i}}^{n} \left| a_{ij}^{(1)} - \frac{a_{1j}^{(1)} a_{i1}^{(1)}}{a_{11}^{(1)}} \right| \leq \sum_{\substack{j=2 \\ j \neq i}}^{n} |a_{ij}^{(1)}| + \sum_{\substack{j=2 \\ j \neq i}}^{n} \left| \frac{a_{1j}^{(1)} a_{i1}^{(1)}}{a_{11}^{(1)}} \right|$$

But since A is strictly diagonally dominant,

$$\sum_{\substack{j=2\\ j\neq i}}^{n} |a_{ij}^{(1)}| < |a_{ii}^{(1)}| - |a_{i1}^{(1)}| \quad \text{and} \quad \sum_{\substack{j=2\\ j\neq i}}^{n} |a_{1j}^{(1)}| < |a_{11}^{(1)}| - |a_{1i}^{(1)}|,$$

so

$$\sum_{\substack{j=2\\ j\neq i}}^{n} |a_{ij}^{(2)}| < |a_{ii}^{(1)}| - |a_{i1}^{(1)}| + \frac{|a_{i1}^{(1)}|}{|a_{11}^{(1)}|}(|a_{11}^{(1)}| - |a_{1i}^{(1)}|) = |a_{ii}^{(1)}| - \frac{|a_{i1}^{(1)}||a_{1i}^{(1)}|}{|a_{11}^{(1)}|}$$

The triangle inequality also implies that

$$|a_{ii}^{(1)}| - \frac{|a_{i1}^{(1)}||a_{1i}^{(1)}|}{|a_{11}^{(1)}|} \leq \left| a_{ii}^{(1)} - \frac{|a_{i1}^{(1)}||a_{1i}^{(1)}|}{|a_{11}^{(1)}|} \right| = |a_{ii}^{(2)}|.$$

which gives

$$\sum_{\substack{j=2 \\ j \neq i}}^{n} |a_{ij}^{(2)}| < |a_{ii}^{(2)}|$$

This establishes the strict diagonal dominance for rows $2, \ldots, n$. But the first row of $A^{(2)}$ and A are the same, so $A^{(2)}$ is strictly diagonally dominant.

The process for other matrices is similar.

Positive Definite Matrices

A matrix A is **positive definite** if it is symmetric and if $\mathbf{x}^t A \mathbf{x} > 0$ for every n-dimensional vector $\mathbf{x} \neq \mathbf{0}$.

$$\mathbf{x}^t A \mathbf{x} = [x_1, x_2, \cdots, x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= [x_1, x_2, \cdots, x_n] \begin{bmatrix} \sum_{j=1}^{n} a_{1j}x_j \\ \sum_{j=1}^{n} a_{2j}x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj}x_j \end{bmatrix} = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_ix_j \right]$$

Example

Show that the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

is positive definite

Solution Suppose $\mathbf{x}$ is any three-dimensional column vector. Then

$$\mathbf{x}^t A \mathbf{x} = [x_1, x_2, x_3] \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1, x_2, x_3] \begin{bmatrix} 2x_1 & - & x_2 & & \\ -x_1 & + & 2x_2 & - & x_3 \\ -x_2 & + & 2x_3 & & \end{bmatrix}$$

$$= 2x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + 2x_3^2.$$

Rearranging the terms gives

$$\begin{aligned}\mathbf{x}^t A\mathbf{x} &= x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + x_3^2 \\ &= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2,\end{aligned}$$

which implies that

$$x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + x_3^2 > 0$$

unless $x_1 = x_2 = x_3 = 0$.

Theorem

If A is an $n \times n$ positive definite matrix, then

(i) A has an inverse; **(ii)** $a_{ii} > 0$, for each $i = 1, 2, \ldots, n$;

(iii) $\max_{1 \le k,j \le n} |a_{kj}| \le \max_{1 \le i \le n} |a_{ii}|$; **(iv)** $(a_{ij})^2 < a_{ii}a_{jj}$, for each $i \ne j$.

Definition

A **leading principal submatrix** of a matrix A is a matrix of the form

$$A_k = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}$$

for some $1 \le k \le n$.

Theorem

A symmetric matrix A is positive definite if and only if each of its leading principal submatrices has a positive determinant.

Example

Show that the following matrix is positive definite,

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solution Note that

$$\det A_1 = \det[2] = 2 > 0,$$

$$\det A_2 = \det \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3 > 0,$$

and

$$\det A_3 = \det\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = 2\det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (-1)\det\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= 2(4-1) + (-2+0) = 4 > 0.$$

In agreement with the previous theorem.

Theorem

The symmetric matrix A is positive definite if and only if Gaussian elimination without row interchanges can be performed on the linear system $A\mathbf{x} = \mathbf{b}$ with all pivot elements positive.

Corollary

LDL^t Factorization

The matrix A is positive definite if and only if A can be factored in the form LDL^t, where L is lower triangular with 1s on its diagonal and D is a diagonal matrix with positive diagonal entries.

Corollary

Let A be a symmetric $n \times n$ matrix for which Gaussian elimination can be applied without row interchanges. Then A can be factored into LDL^t, where L is lower triangular with 1s on its diagonal and D is the diagonal matrix with $a_{11}^{(1)}, \ldots, a_{nn}^{(n)}$ on its diagonal.

Example

Determine the LDL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}$$

Solution The LDL^t factorization has 1s on the diagonal of the lower triangular matrix L so we need to have

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} 1 & l_{21} & l_{31} \\ 0 & 1 & l_{32} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 & d_1 l_{21} & d_1 l_{31} \\ d_1 l_{21} & d_2 + d_1 l_{21}^2 & d_2 l_{32} + d_1 l_{21} l_{31} \\ d_1 l_{31} & d_1 l_{21} l_{31} + d_2 l_{32} & d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \end{bmatrix}$$

Thus

$$a_{11} : 4 = d_1 \Longrightarrow d_1 = 4, \qquad a_{21} : -1 = d_1 l_{21} \Longrightarrow l_{21} = -0.25$$

$$a_{31} : 1 = d_1 l_{31} \Longrightarrow l_{31} = 0.25, \qquad a_{22} : 4.25 = d_2 + d_1 l_{21}^2 \Longrightarrow d_2 = 4$$

$$a_{32} : 2.75 = d_1 l_{21} l_{31} + d_2 l_{32} \Longrightarrow l_{32} = 0.75, \quad a_{33} : 3.5 = d_1 l_{31}^2 + d_2 l_{32}^2 + d_3 \Longrightarrow d_3 = 1,$$

and we have

$$A = LDL^t = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.25 & 0.75 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.25 & 0.25 \\ 0 & 1 & 0.75 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \blacksquare$$

Corollary

Cholesky Factorization

The matrix A is positive definite if and only if A can be factored in the form LL^t, where L is lower triangular with nonzero diagonal entries.

Example

Determine the Cholesky LL^t factorization of the positive definite matrix

$$A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 4.25 & 2.75 \\ 1 & 2.75 & 3.5 \end{bmatrix}$$

Solution The LL^t factorization does not necessarily has 1s on the diagonal of the lower triangular matrix L so we need to have

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Thus

$a_{11}: \quad 4 = l_{11}^2 \implies l_{11} = 2,$ $\qquad a_{21}: \quad -1 = l_{11}l_{21} \implies l_{21} = -0.5$

$a_{31}: \quad 1 = l_{11}l_{31} \implies l_{31} = 0.5,$ $\qquad a_{22}: \quad 4.25 = l_{21}^2 + l_{22}^2 \implies l_{22} = 2$

$a_{32}: \quad 2.75 = l_{21}l_{31} + l_{22}l_{32} \implies l_{32} = 1.5,$ $\qquad a_{33}: \quad 3.5 = l_{31}^2 + l_{32}^2 + l_{33}^2 \implies l_{33} = 1,$

and we have

$$A = LL^t = \begin{bmatrix} 2 & 0 & 0 \\ -0.5 & 2 & 0 \\ 0.5 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -0.5 & 0.5 \\ 0 & 2 & 1.5 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \blacksquare$$

Band Matrices

Matrices that all their nonzero entries concentrate about the diagonal.

band width $\qquad w = p + q - 1$

p: the number of diagonals above, and including, the mail diagonal on

which nonzero entries may lie.

q : the number of diagonals below, and including, the mail diagonal on

which nonzero entries may lie.

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & -5 & -6 \end{bmatrix}$$

$p = q = 2$

w = 2 + 2 - 1 = 3

Tridiagonal Matrices

$p = q = 2$ Bandwidth = 3

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & & \vdots \\ 0 & a_{32} & a_{33} & a_{34} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_{n-1,n} \\ 0 & \cdots & 0 & & a_{n,n-1} & a_{nn} \end{bmatrix}$$

Crout Factorization for Tridiagonal Linear Systems

$A = LU$

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & l_{n,n-1} & l_{nn} \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & u_{12} & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & u_{n-1,n} \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

The multiplication involved with $A = LU$ gives,

$$a_{11} = l_{11};$$

$$a_{i,i-1} = l_{i,i-1}, \quad \text{for each } i = 2, 3, \ldots, n;$$

$$a_{ii} = l_{i,i-1}u_{i-1,i} + l_{ii}, \quad \text{for each } i = 2, 3, \ldots, n;$$

$$a_{i,i+1} = l_{ii}u_{i,i+1}, \quad \text{for each } i = 1, 2, \ldots, n-1.$$